Algebra-I
Backpaper Exam
B. Math - First year
2014-2015

Time: 3 hrs
Max score: 100

Answer all questions.

(1) (a) Show that S_n is generated by $(1\ 2)$ and $(1\ 2\ 3 \ldots n)$ for all $n \geq 2$.
(b) Can $(1\ 2)$ be replaced by any transposition? Give reasons or counter example to support your answer. 8+8

(2) (a) Let G be a group and H be a subgroup of G. Consider the action of G on the left cosets of H in G by left multiplication. Determine the kernel of the action and show that the kernel is the largest normal subgroup of G contained in H.
(b) Prove that if H has finite index n then there is a normal subgroup K of G, $K \subseteq H$, such that $|G : K| \leq n!$. 8+6

(3) (a) Prove that two elements of S_n are conjugate if and only if they have the same cycle type.
(b) Determine the elements of $C_{S_7}(\sigma)$ where $\sigma = (1\ 4\ 5)$. 8+8

(4) (a) Find all finite groups which have exactly two conjugacy classes.
(b) Show that if n is odd then the set of all n-cycles consists of two conjugacy classes of equal size in A_n. 6+8

(5) (a) Show that $G/Z(G)$ is isomorphic to a subgroup of the automorphism group $Aut(G)$.
(b) Let G be a group of order 203. Prove that if H is a normal subgroup of order 7 in G then $H \subseteq Z(G)$. Deduce that G is abelian in this case. 8+8

(6) (a) State Sylow's theorems.
(b) Prove that a group of order 12 either contains a normal Sylow 3-subgroup or is isomorphic to the alternating group A_4. 5+8

(7) (a) Show that if $o(G) = 60$ and G has more than one Sylow 5-subgroup, then G is simple.
(b) Hence show that A_5 is simple. 8+3

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